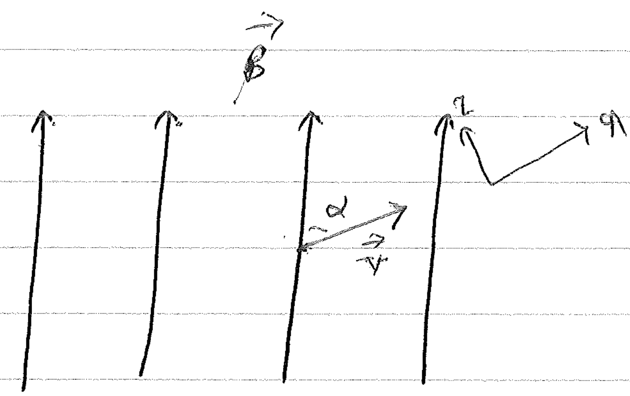
Lec 14: 03/05/2012

Synchrothron Radiation:

A non-relativistic electron's motion in the presence of a magnetic field $\vec{B}$ is a superposition of a translational path with constant velocity $v_{\parallel} = v\cos\alpha$ and a circular accelerated component with velocity $v_{\perp} = v\sin\alpha$:



$$m_e \frac{v_\perp^2}{r_{gyr}} = \frac{e v B \sin\alpha}{c} \Rightarrow$$

$$r_{gyr} = \frac{v \sin\alpha}{\omega_{gyr}}$$

$$\omega_{gyr} \equiv \frac{eB}{m_e c} \simeq 1.8 \times 10^7 \left(\frac{B}{1G}\right)$$

The power emitted by the electron is given by Larmor's equation:

$$P = \frac{2e^2}{3c^3} \omega_{gyr}^2 \, v^2 \sin^2\alpha$$

It emerges as monochromatic radiation with angular frequency ω_{gyr}. It is known as "Cyclotron Radiation". The monochromatic spectrum associated with cyclotron radiation is essentially independent of the viewing angle. This situation will change dramatically when $v \to c$.

In the relativistic case, it will be more convenient to move to the rest frame of the electron. In this frame the $\vec{E}$ and $\vec{B}$ field components are:

$$E^{x\prime} = E^{x} \quad , \quad E^{y\prime} = \gamma\left(E^{y} - \frac{v}{c}B^{z}\right) \quad , \quad E^{z\prime} = \gamma\left(E^{z} + \frac{v}{c}B^{y}\right)$$

In the lab frame $\vec{E} = 0$, $B^{y} = 0$. Therefore;

$$E^{x\prime} = 0 \quad , \quad E^{y\prime} = -\frac{\gamma v}{c} B\sin\alpha \quad , \quad E^{z\prime} = 0$$

Since electron is at rest in its rest frame, we have;

$$\vec{a}' = \frac{e\gamma v B\sin\alpha}{m_e c} = \gamma v \omega_{gyr} \sin\alpha$$

The power radiated by the electron in its rest frame is;

$$P' = \frac{2e^2}{3c^3}\gamma^2 \omega_{gyr}^2 v^2 \sin^2\alpha = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B$$

Here σ_T is the Thomson scattering cross-section and $U_B = \frac{B^2}{8\pi}$ is the energy density in the magnetic field.

As pointed out before, the radiated power is a Lorentz-invariant quantity, which implies that in the lab frame we have:

$P_{Sync} = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 u_B$

Note the similarity between P_{Sync} and P_{Comp} that we derived previously ($P_{Comp} = \frac{4}{3} \sigma_T c\beta^2 \gamma^2 u_{rad}$). The similarity between P_{Sync} and P_{Comp} is not a coincidence. Both of the Synchrotron and Compton emissions are based on the same physical interaction, which is the collision between a charged particle and a photon. Thus, the power emitted by the charged particle should depend only on the density of photons. In the case of Compton emission, the photon density is $u_{rad} = \frac{E^2}{4\pi}$, while in the case of Synchrotron radiation the photon density is $u_B = \frac{B^2}{8\pi}$.

Next, consider the frequency dependence of Synchrotron radiation.

The radiation pattern in the electron's rest frame is that of a dipole. Note that the dipole approximation is valid in this frame because the velocity is just zero. It will look very different in the lab frame though, because $v \simeq c$ in this frame.

The transformation law for the angle is:

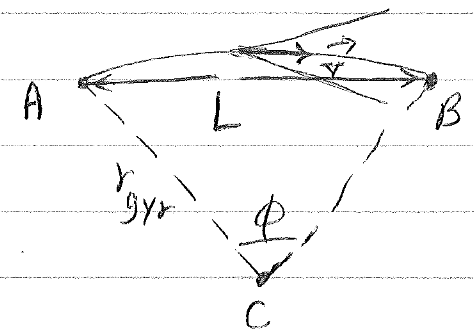
$$\sin\psi = \frac{\sin\psi'}{\gamma(1+\beta\cos\psi')}$$

For $\beta \simeq 1$ and $\gamma \gg 1$, this results in:

$$\sin\psi \simeq \psi \simeq \frac{1}{\gamma}$$

Thus, whereas the power is radiated nearly isotropically in the electron's rest frame, most of it is beamed into a narrow cone with half-opening angle $\sim \frac{1}{\gamma}$ in the lab frame.

Because of this beaming, the emission is pulsed every time the cone sweeps around the line of sight. Another important point has to do with the actual interval over which the observer sees radiation.



$$t_{AB} = \frac{L}{v}$$

$$t^{o}_{AB} = \left(1 - \frac{v}{c}\right) t_{AB}$$

Here t^{o}_{AB} is the arrival time. $\phi = \frac{2}{\gamma}$ from the above discussion,

which results in:

$$t^{o}_{AB} \doteq \left(1 - \frac{v}{c}\right) \frac{2}{\gamma} \frac{1}{\omega^{rel}_{gyr}}$$

Note that the relativistic angular gyration frequency ω^{rel}_{gyr} is different from ω_{gyr} mentioned before. It can be calculated from the electron's equation of motion:

$$\frac{d}{dt}(\gamma m_e \vec{v}) = \frac{e}{c} \vec{v} \times \vec{B}$$

Since $\vec{E} = 0$ in the lab frame, we see that:

$$\frac{d}{dt}(\gamma m_e c^2) = e \vec{v} \cdot \vec{E} = 0 \Rightarrow \gamma = \text{const.}$$

Thus:

$$m_e \gamma \frac{d\vec{v}}{dt} = \frac{e}{c} \vec{v} \times \vec{B} \Rightarrow \frac{d\vec{v}_\perp}{dt} = \frac{e}{\gamma m_e c} \vec{v}_\perp \times \vec{B} \Rightarrow$$

$$\omega_{gyr}^{rel} = \frac{eB}{\gamma m_e c}$$

This is exactly $\frac{\omega_{gyr}}{\gamma}$, with γ being due to time dilation. We therefore find,

$$t_{AB}^{o} \simeq \frac{1}{\gamma^2 \omega_{gyr}}$$

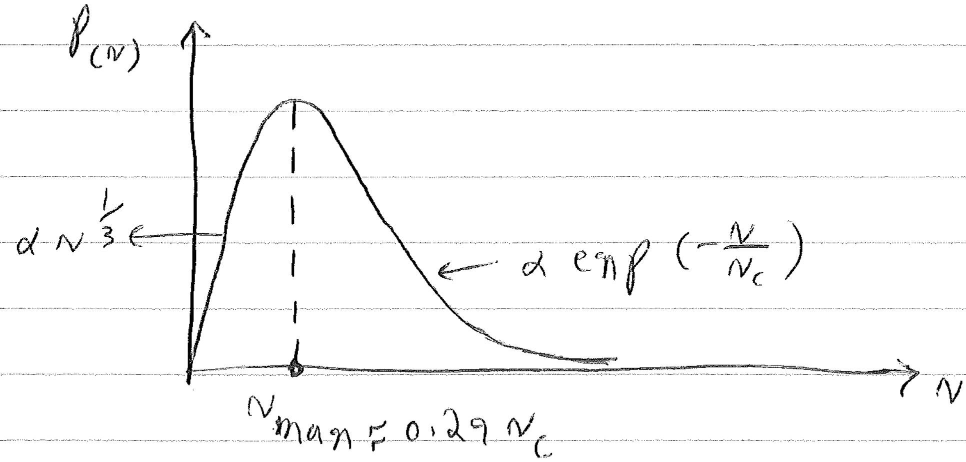
The relativistic arrival time is much shorter than the orbital period. Hence the dominant frequency of radiation $\sim (t_{AB}^{o})^{-1}$ is much higher than ν_{gyr}^{rel}. In addition, because $\omega_{gyr}^{rel} \ll \omega_{gyr}$, the spacing between adjacent frequencies is much smaller than in the non-relativistic case. As a result, many discrete emission lines blend together near $\gamma^2 \nu_{gyr}$ to form a continuum in the relativistic case. Taking all the Fourier components into account results in the following expression for the total power per unit frequency:

$$P_{(\nu)}\, d\nu = \frac{\sqrt{3}\, e^3 B \sin\alpha}{m_e c^2} \frac{\nu}{\nu_c}\, d\nu \int_{\nu/\nu_c}^{\infty} K_{\frac{5}{3}}(\eta)\, d\eta$$

Here $K_{\frac{5}{3}}$ is the Bessel function of order $\frac{5}{3}$ and:

$$\nu_c \equiv \frac{3}{2} \gamma^2 \nu_{gyr} \sin\alpha$$

The spectrum of synchrothron radiation by a single electron looks like as follows:



The use of the above equation for $P_{(\nu)}$ is rather limited. Synchrothron radiation from astrophysical sources is typically produced by an ensemble of particles. This removes the pitch angle α from the expression, for an isotropical distribution, and one has to consider an energy distribution for electrons. The total power radiated by an ensemble as a function of frequency is given by:

$$P_{tot}(\nu) = \int_{E_1}^{E_2} P_{(\nu)} \, N_{(E)} \, dE$$

First consider thermal Synchrothron radiation. In this case:

$$N_{(E)} \, dE = N_0 E^2 \exp\left(\frac{-E}{kT}\right) dE \qquad (E \gtrsim m_e c^2)$$

Then:

$$P_{tot}(\nu) \approx \frac{\sqrt{3}\, e^3 n_e B}{8\pi m_e c^2} \left(\frac{\nu}{\nu_T}\right) I\left(\frac{\nu}{\nu_T}\right)$$

Here:

$$\nu_T \equiv \frac{3eB(kT)^2}{4\pi m_e^3 c^5}, \qquad I_{(x)} \equiv \frac{1}{x} \int_0^\infty u^2 e^{-u} F\left(\frac{x}{u^2}\right) du$$

Where:

$$F_{(y)} \equiv y \int_y^\infty K_{5/3}(u) \, du$$

$I_{(x)}$ can be approximated as a power law over most of its range, with different indices above and below ~ 1.

In general, thermal processes are not very efficient unless th plasma temperature is very high ($T \gg \frac{m_e c^2}{k}$). Therefore, in most cases thermal synchrothron sources are too faint to be seen easily. This situation has changed in recent years as the instrument sensitivity has continued to improve. For example, the supermassive black hole at the center of the galaxy is an excellent example of an object whose spectrum includes a thermal synchrothron component, which has emerged as an important high-energy source in recent years.

Most synchrothron sources, including the relativistic jets in AGNs, and the shells in supernova remnants, are non-thermal sources. In these sources electrons are accelerated to relativistic velocities. Their distribution is given by a power law in this case.

$$N(E)\,dE = KE^{-\alpha}\,dE$$

Here K is a normalization constant and the spectral index α

typically varies over the range $\sim 2 - 2.5$.

Recall that for a given electron energy E most of the radiated power is at $\nu_c(E)$:

$$\nu_c(E) \simeq \left(\frac{E}{m_e c^2}\right)^2 \nu_{gyr} \qquad , \qquad \nu_{gyr} \equiv \frac{eB}{2\pi m_e c}$$

This implies that:

$$dE \simeq \frac{m_e c^2}{2(\nu_{gyr}\nu)^{1/2}} \, d\nu$$

Since:

$$P = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B \simeq \frac{4}{3}\sigma_T c \left(\frac{E}{m_e c^2}\right)^2 \frac{B^2}{8\pi} \qquad (\beta \simeq 1)$$

We find:

$$P_{tot}(\nu) = \int P \, N(E) \, dE \propto B^{\frac{1+n}{2}} \, \nu^{\frac{1-n}{2}}$$

The complete expression is:

$$P_{tot}(\nu) = 1.7 \times 10^{-21} \, a(n) \, K B^{\frac{1+n}{2}} \times \left(\frac{6.26 \times 10^{18}\,\mathrm{Hz}}{\nu}\right)^{\frac{n-1}{2}} \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}$$

$a(n)$ is a function that depends only weakly on the spectral index.

The most important property of non-thermal synchrotron

radiation is the power-law spectrum with index $\frac{n-1}{2}$.
However, one should note that it is not valid over all frequencies.
Indeed, the spectrum is divergent at low frequencies. But exceeding the blackbody limit is not permitted on physical grounds. In fact, thermodynamics considerations tell us that the emissivity of a plasma cannot exceed the limit imposed by the maximal rate at which the plasma can absorb energy, which constitutes a blackbody limit.

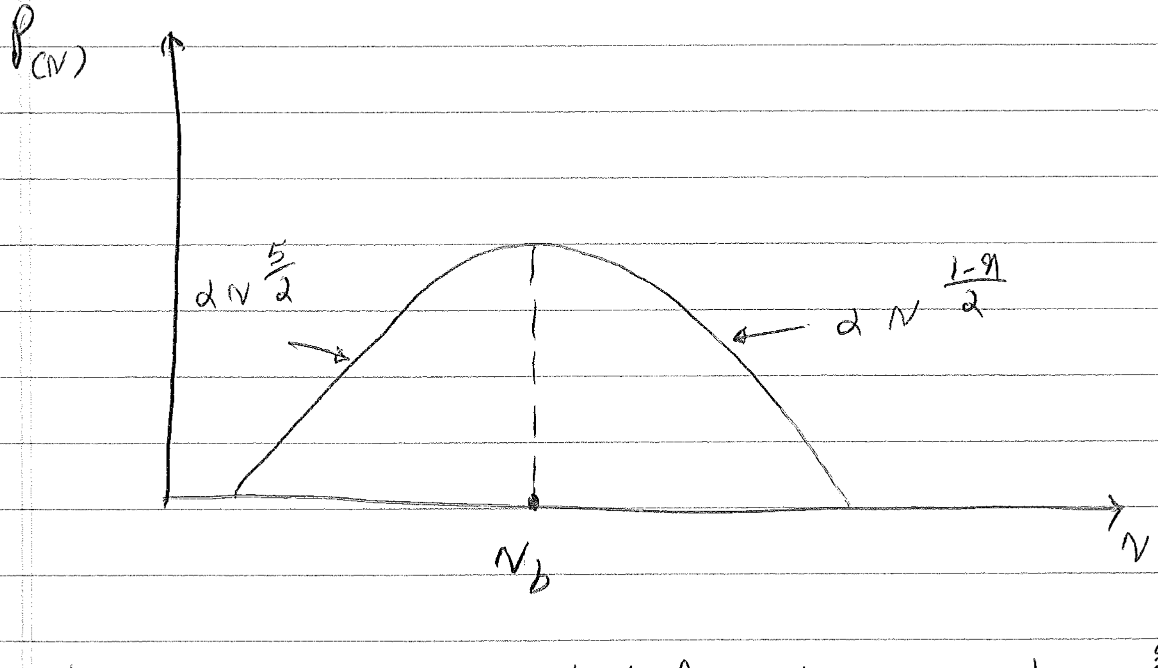
A quick solution to the divergence of $P_{tot}(\nu)$ at low frequencies would be that the non-thermal synchrothron spectrum should be a power law except below some break frequency ν_b. The spectrum would go $\propto \nu^2 T$ below ν_b, similar to the Rayleigh-Jeans portion of the blackbody spectrum. T is an effective temperature, which can be frequency-dependent, rather than an actual temperature that exists for a thermal distribution. It is related to the energy of the electrons, which results in;

$$\gamma m_e c^2 \simeq \frac{3}{2} k T_{(\nu)} \Rightarrow T_{(\nu)} \sim \frac{m_e c^2}{3k} \gamma \sim \frac{m_e c^2}{3k} \left(\frac{\nu}{\nu_{gyr}} \right)^{\frac{1}{2}}$$

The synchrothron spectrum below ν_b therefore goes as;

$$P_{tot}(\nu) \propto \nu^{\frac{5}{2}}$$

The non-thermal synchrothron spectrum therefore looks as follows;



This spectrum is unique and informative. The $\nu^{\frac{5}{2}}$ behavior at low frequencies is a clear indication that the radiation is non-thermal. The slope at high frequencies provides us with the spectral index of the emitting particles. Finally, the break frequency ν_b is a measure of the optical depth through emitter.

Let us close our discussion of Synchrothron radiation by a comment. The similarity of P_{sync} and P_{comp} provides us with a powerful tool in probing the physical conditions at the high-energy object. For example, if we can measure both a radio and a γ-ray spectrum in the same object, then the ratio $\frac{P_{sync}}{P_{comp}}$ allows us to infer the magnetic field if we know the local U_{rad}, or to probe the ambient photon intensity if we have a measure of the magnetic field (for example, from Faraday rotation measurements).